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On a Table for the Formation of Logarithms and Anti-Logarithms to Twelve Places. By PETER GRAY, F.R.A.S., *Honorary Member of the Institute of Actuaries.* (*Part II.*)

[Read before the Institute, 30th January, 1865.]

IN the last number of the *Journal* I explained and exemplified a method for the formation of logarithms and anti-logarithms to twelve places, and gave the tables requisite for its application. I am now to analyse and further exemplify the method in question, and demonstrate the rules laid down in the previous paper.

Let N be any given number, whose logarithm to t places is required.

Divide or multiply N by a, any number that will give a result having unit for its first figure. Place the decimal point after this first figure, and denote the result so modified by $1+N^1$, in which N^1 consequently will be a decimal fraction. Then, denoting by m the number of places through which the decimal point was moved (*plus* or *minus*, according as the movement was to the left or to the right), we should have

$$N=10^{\pm m}.a^{\pm 1}(1+N^1) \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

Let $1+n_1$ denote *so many* (it is no matter for the present purpose *how* many) of the leading figures of $1+N^1$; subtract $1+n_1$ from $1+N^1$, and let the remainder be r_1; divide r_1 by $1+n_1$, and let the quotient and the remainder be n_2 and r_2 respectively; divide r_2 by $(1+n_1)(1+n_2)$, and let the quotient and the remainder of this division be n_3 and r_3 respectively. Proceeding thus, we find that division of r_{p-1} by $(1+n_1)(1+n_2)\ldots(1+n_{p-1})$ gives for quotient and remainder n_p and r_p respectively.

This operation may be typically represented as follows:—

$$\begin{array}{rl}
 & 1+N^1 \\
 & \underline{1+n_1} \\
1+n_1\,\big) & r_1\ \ (\,n_2 \\
 & \underline{(1+n_1)n_2} \\
(1+n_1)(1+n_2)\,\big) & r_2\ \ (\,n_3 \\
 & \underline{(1+n_1)(1+n_2)n_3} \\
 & r_3 \\
 & \vdots \\
(1+n_1)(1+n_2)\ldots(1+n_{p-1})\,\big) & r_{p-1}\ \ (\,n_p \\
 & \underline{(1+n_1)(1+n_2)\ldots(1+n_{p-1})n_p}\,. \\
 & r_p
\end{array}$$

From this we obviously have,

$$\begin{array}{rcl}
r_1 & = & (1+N^1)-(1+n_1) \\
r_2 & = & r_1-(1+n_1)n_2 \\
r_3 & = & r_2-(1+n_1)(1+n_2)n_3 \\
\vdots & & \vdots \\
r_{p-1} & = & r_{p-2}-(1+n_1)(1+n_2)\ldots(1+n_{p-2})n_{p-1} \\
r_p & = & r_{p-1}-(1+n_1)(1+n_2)\ldots(1+n_{p-1})n_p;
\end{array}$$

or, by transposition,

$$\begin{array}{rcl}
(1+N^1)-r_1 & = & (1+n_1) \\
r_1-r_2 & = & (1+n_1)n_2 \\
r_2-r_3 & = & (1+n_1)(1+n_2)n_3 \\
\vdots & & \vdots \\
r_{p-2}-r_{p-1} & = & (1+n_1)(1+n_2)\ldots(1+n_{p-2})n_{p-1} \\
r_{p-1}-r_p & = & (1+n_1)(1+n_2)\ldots(1+n_{p-1})n_p.
\end{array}$$

Now, the foregoing equations may be written as follows:—

$$\begin{array}{rcl}
(1+N^1)-r_1 & = & (1+n_1) \\
r_1-r_2 & = & (1+n_1)(1+n_2)-(1+n_1) \\
r_2-r_3 & = & (1+n_1)(1+n_2)(1+n_3)-(1+n_1)(1+n_2) \\
\vdots & & \vdots \\
r_{p-2}-r_{p-1} & = & (1+n_1)(1+n_2)\ldots(1+n_{p-1})-(1+n_1)(1+n_2)\ldots(1+n_{p-2}) \\
r_{p-1}-r_p & = & (1+n_1)(1+n_2)\ldots(1+n_p)-(1+n_1)(1+n_2)\ldots(1+n_{p-1}).
\end{array}$$

Adding together these last equations, we obtain,

$$\left.\begin{array}{l}(1+N^1)-r_p=(1+n_1)(1+n_2)(1+n_3)\ldots(1+n_p) \\ \text{or,} \\ 1+N^1=(1+n_1)(1+n_2)(1+n_3)\ldots(1+n_p)+r_p\end{array}\right\} \quad . \ (2).$$

From this it appears that, since the remainders r_2, r_3, &c., successively decrease, the product $(1+n_1)(1+n_2)\ldots$ constantly approximates to $1+N^1$; so that, if the operation above described and typified shall have been continued till r_p has no significant figures in the first t decimal places, the equation

$$1+N^1=(1+n_1)(1+n_2)(1+n_3)\ldots(1+n_p) \quad . \ . \ . \quad (3)$$

will be true to t decimal places, or to $t+1$ places in all.

We may now substitute this value of $1+N^1$ in equation (1), which thus becomes

$$N=10^{\pm m}.a^{\pm 1}(1+n_1)(1+n_2)\ldots(1+n_p) \quad . \ . \ . \ . \quad (4).$$

Now, we know that the first t places of a logarithm are not affected by any figure beyond the $(t+1)$th place of the corresponding number;* hence, equation (4) being true to $t+1$ places, if we take the logarithms of both sides, the resulting equation will be true to t places. So we should have

$$\log N=\pm m\pm\log a+\log(1+n_1)+\log(1+n_2)\ldots+\log(1+n_p) \ . \ (5).$$

This expression admits of certain modifications, by which the facility of its application is increased.

First. By inspection of any given number we can at once assign the index of its logarithm. We may, therefore, neglect all indices in the course of the operation; and hence m, which, being an integer, affects only the index of the logarithm, may be omitted from the formula.

Secondly. If for $-\log a$, when it occurs, we write $\text{colog}\, a$, the component logarithms will then always have to be combined by addition.

We thus obtain for the determination of the mantissa of log N, the two following formulæ :—

$$\log a+\log(1+n_1)+\log(1+n_2)\ldots+\log(1+n_p),$$

and

$$\text{colog}\, a+\log(1+n_1)+\log(1+n_2)\ldots+\log(1+n_p);$$

---

* This can easily be shown by aid of the expression

$$\log(x+h)-\log x=M\left\{\frac{h}{x}-\frac{h^2}{2x^2}+\ldots\right\},$$

where M is the modulus, =·43429 . . . . And it is for the reason above stated that the decimal places in $1+N^1$ are, in the resolving process, restricted to t, the number of places there are to be in the mantissa of the required logarithm.

the first for use when a has been employed as a *divisor*, and the second when it has been employed as a *multiplier*.

The operation which has now been symbolically described and illustrated will, perhaps, hardly as yet have been recognised as identical with that which was employed in the previous paper for the formation of logarithms. The arithmetical illustrations I am now about to give will, however, serve to establish this identity.

For the application of these formulæ we must be provided with, first, a table of $\log a$ and $\operatorname{colog} a$; which, as a needs never be a number consisting of more than a single digit, will be simply a table of the logarithms and co-logarithms of the numbers 2 to 9. And secondly, we require a table of $\log(1+n)$ for all the values of n that can arise in the course of the operation. The extent of this table will depend mainly on the number of figures in n_1, n_2, &c.; that is, on the extent to which we carry the several divisions in the resolving process. By the terms of the preceding investigation we are laid under no restrictions in this respect, but are left at liberty to adopt the course which may seem most expedient. It is soon found that every increase in the number of figures in n_1, n_2, &c., besides possessing other advantages, materially simplifies the resolving process, while at the same time the extent of the table of $\log(1+n)$ is largely increased. A compromise, therefore, becomes necessary, which shall allow us to enjoy the greatest amount of facility in the process that may be consistent with a moderate extent of the table. I believe that, for twelve-figure logarithms, the course I have adopted is the best. I carry the divisions to three figures, and the table consists of 3,000 values, which admit of very lucid arrangement in ten pages. Had I gone a step further, and used four-figure quotients, a table consisting of 30,000 values would have been necessary.

In the following example, illustrative of the resolving process, I shall give one, two, and three-figure solutions. We shall thus be more naturally conducted from the form of the process typified above to the somewhat modified form of it employed in the previous paper. The advantage of the three-figure process over the others will thus also be rendered more apparent.

Resolve $\pi = 3{\cdot}14159{,}26535{,}90$, into factors.

The operation by the one-figure process is as follows :—

	3141592653590	÷2		3141592653590	×4
1·5	~~1~~·~~5~~70796326795	4	1·2	~~1~~·~~2~~56637061436	4
60	60			48	
1·560	)10796	6	1·248	)8637	6
9360	9360			7488	
1·569360	)14363267	9	1·255488	)11490614	9
141242	14124240			11299392	
1·5707724~~2~~	)23902795	1	1·2566179~~4~~	)19122236	1
1571	15707724			12566179	
1·570788~~1~~	)8195071	5	1·256630~~6~~	)6556057	5
79	7853941			6283153	
1·57079~~6~~	)341130	2	1·25663~~7~~	)272904	2
	314159			251327	
	26971	1		21577	1
	15708			12566	
	11263	7		9011	7
	10996			8796	
	267	1		215	1
	157			126	
	110	7		89	7
	110			88	
	0	0		1	1

Two solutions are given, on which I shall remark in order. Comparing the process on the left with the type on page 122, it is seen that 2 is taken for a, and division gives for $1 + N^1$, 1·5707 . . . The leading unit and the first decimal place of this, namely, 1·5, are taken for $1+n_1$. Instead of formally subtracting this from $1+N^1$, as in the type, the figures are struck out, and re-written a little to the left. Division of $r_1 = {\cdot}07079 \ldots$ by $1+n_1$, gives $\cdot 04 = n_2$. We now form a new divisor. This is $(1+n_1)(1+n_2)$, and it is formed by adding to $(1+n_1)$ its product by n_2, which product is got from the adjoining column. The divisor having thus received an accession of two figures, we must, to restore the relation between divisor and remainder, annex two figures to the remainder of the previous division; and taking down another figure, we obtain by division another quotient, $\cdot 006 = n_3$. Forming a new divisor as before, by adding to the last the product opposite in the adjoining column, we find it contains three figures more than the preceding;

and therefore three figures have to be taken down to the remainder to restore the relation, and one figure more to allow the division to go on. This gives us $n_4 = {\cdot}0009$. At this point we see that there remain only two figures to be taken down from the dividend, and it is therefore unnecessary to extend the divisor more than two places. Hence, only so much of the addend for the new divisor as will give an accession of two places is taken from the adjoining column, and the last two figures of the dividend being annexed to the remainder, the divisor is curtailed of its last figure to allow the division to proceed. This gives us $n_5 = {\cdot}00001$. The next two divisors are in like manner each curtailed of a figure; and when the last of them (the sixth) is reached, it is seen that it is unnecessary to form any more, as from that point the effective figures would undergo no change. The operation, therefore, passes into that of contracted division.

The above process may be described as one of division by a variable divisor; the object and the effect of the arrangement for restoring the relation between divisor and remainder after each accession to the former being, to preserve the position in the scale which belongs to them of the several quotient figures, which is that of orderly succession from the decimal point. Writing them thus, commencing with n_1,

$$\cdot546915217170,$$

each occupies its proper position with respect to the decimal point.

It will be observed that the sixth divisor agrees, so far as it is formed, with the figures of $1+N^1$; and if the formation were extended as far as twelve decimal places, and the succeeding products in the adjoining column added to it, the sum would be found equal to $1+N^1$. This is in accordance with equation (3).

The divisors admit of being formed in a different manner, by which the writing of a number of figures is saved.

By equation (2),

$$(1+n_1)(1+n_2)\ldots(1+n_p)=(1+N^1)-r_p.$$

Now the left-hand member of this equation is, by the type, the pth divisor, and r_p is the remainder at the same point. Hence, each new divisor may be formed by subtracting the remainder, at the point of the operation attained, from the figures of $1+N^1$ directly over it. Thus, the three subtractions following give the second, third, and fourth divisors :—

$$\begin{array}{r} 1{\cdot}570 \\ 10 \\ \hline 1{\cdot}560 \end{array} \qquad \begin{array}{r} 1{\cdot}570796 \\ 1436 \\ \hline 1{\cdot}569360 \end{array} \qquad \begin{array}{r} 1{\cdot}5707963267 \\ 239027 \\ \hline 1{\cdot}5707724240 \end{array}$$

These subtractions can be performed by inspection, and the results at once set down in their places.

In the resolution on the right, in which 4, used as a multiplier, is taken for *a*, the last method of forming the divisors is employed. Its advantage over the other will be more manifest in the case of the two and three-figure solutions.

The two sets of factors into which, in accordance with (4), π is resolved by the preceding processes, are—

2 × 1·5	and ¼ × 1·2
× 1·04	× 1·04
× 1·006	× 1·006
× 1·0009	× 1·0009
× 1·00001	× 1·00001
× 1·000005	× 1·000005
× 1·0000002	× 1·0000002
× 1·00000001	× 1·00000001
× 1·000000007	× 1·000000007
× 1·0000000001	× 1·0000000001
× 1·00000000007	× 1·00000000007
× 1·000000000000;	× 1·000000000001.*

From the form of these factors it appears that the table requisite for the application of the one-figure process must be one of $\log(1 + {\cdot}1^{m}n)$, for values of n from 0 to 9, and values of m from 1 to 12. These logarithms admit of convenient arrangement in twelve columns, corresponding to the values of m, an additional column receiving the values of n, to serve as argument. By the aid of such a table, $\log \pi$, as arising from the two preceding operations, is formed as follows:—

301029995664	log 2		397940008672	colog 4	
176091259056	5	Col. 1	79181246048	2	Col. 1
17033339299	4	„ 2	17033339299	4	„ 2
2597980720	6	„ 3	2597980720	6	„ 3
390689250	9	„ 4	390689250	9	„ 4
4342923	1	„ 5	4342923	1	„ 5
2171467	5	„ 6	2171467	5	„ 6
86859	2	„ 7	86859	2	„ 7
4343	1	„ 8	4343	1	„ 8
3040	7	„ 9	3040	7	„ 9
43	1	„ 10	43	1	„ 10
30	7	„ 11	30	7	„ 11
0	0	„ 12	0	1	„ 12
0·497149872694	log π		0·497149872694	log π	

---

* It will be observed that the last eleven factors are the same in both sets. This arises from the *casual* circumstance that the figures of the products of the first two factors are the same in both: 2 × 1·5 = 3, and ¼ × 1·2 = ·3.

Both results are true in the last figure.

I now propose the resolution of the same number, π, by the two-figure process.

The operation follows:—

	3 14 15 92 65 35 90	÷3	31 41 59 26 53 59 0	×6
1·04	~~1·04~~ 71 97 55 11 97	69	1·88)49 55 59 21 54	26
62 4	62 4		37 6	
	9 57		11 95	
9 36	9 36		11 28	
1·04 71 76	)21 55 11 9	20	1·88 48 88)67 59 21 5	35
21	20 94 35 2		56 54 66 4	
1·04 71 9~~7~~	)60 76 77	58	11 04 55 14	
	52 35 99		9 42 44 40	
	8 40 78		1·88 49 5~~4~~)1 62 10 74	86
	8 37 76		1 50 79 63	
	3 02	02	11 31 11	
	2 09		11 30 97	
	93	89	14	00 07
	84		13	
	9			

Two resolutions are given, as before, using different preparing factors, and forming the divisors in the second resolution by subtraction. These operations do not require the same minuteness of description as the former. In consequence of the extension of the quotients to two figures, only three divisors have to be formed, and the operations then, as before, pass into contracted division. The two sets of factors are :—

$3 \times 1{\cdot}04$	and $\frac{1}{6} \times 1{\cdot}88$
$\times 1{\cdot}00{,}69$	$\times 1{\cdot}00{,}26$
$\times 1{\cdot}00{,}00{,}20$	$\times 1{\cdot}00{,}00{,}35$
$\times 1{\cdot}00{,}00{,}00{,}58$	$\times 1{\cdot}00{,}00{,}00{,}86$
$\times 1{\cdot}00{,}00{,}00{,}00{,}02$	$\times 1{\cdot}00{,}00{,}00{,}00{,}00{,}07.$
$\times 1{\cdot}00{,}00{,}00{,}00{,}00{,}89;$	

The orderly succession of the quotient figures from the decimal point is here seen to be the same as in the former case. And hence also, the table requisite for use in connexion with this process will be one of $\log(1 + {\cdot}01^m n)$, for values of n from 00 to 99, and values of m from 1 to 6. It will thus occupy six columns of 100 values each. The following shows the application of such a table to the above resolutions:—

47 71 21 25 47 20	log 3	22 18 48 74 96 16	colog 6
1 70 33 33 92 99	04 Col. 1	27 41 57 84 92 64	88 Col. 1
29 86 34 08 57	69 „ 2	11 27 70 02 77	26 „ 2
8 68 58 03	20 „ 3	15 20 00 41	35 „ 3
25 18 91	58 „ 4	37 34 93	86 „ 4
87	02 „ 5	00	00 „ 5
39	89 „ 6	3	07 „ 6
0·49 71 49 87 26 96		0·49 71 49 87 26 94	

Finally, I propose the resolution of π into factors by the three-figure process.

This is the process with which we were familiarised in the former paper. I resolve the given number in two ways, as follows:—

	3·141 592 653 590	÷2	314 159 265 359 0	×6
1·570	$\cancel{1}\cdot\cancel{5}7\cancel{0}$ 796 326 795	507	1·884)955 592 154	507
785	785 0		942 0	
	11 326		13 592	
11	10 990		13 188	
1·570 79$\cancel{6}$	)336 795	214	1·884 95$\cancel{5}$)404 154	214
	314 159		376 991	
	22 636		27 163	
	15 708		18 850	
	6 928		8 313	
	6 283		7 540	
	645	410	773	410
	628		754	
	17		19	
	16		19	
	1		0	

Little needs now to be said in explanation. The number of factors in $1+N^1$ is reduced to four, and the number of divisors to two. As there remain, after the formation of n_2 ($=$·000,507), only three figures to be annexed to the remainder, it is unnecessary to make provision for an increase of more than the same number in the figures of the divisor. The division then goes on to the close by periodical curtailment of the divisor.

Referring to the right-hand operation, the complete second divisor would be formed as follows:—

$$\begin{array}{r} 1\cdot884 \\ 942 \\ 13\ 188 \\ \hline 1\cdot884{,}955{,}188 \end{array} \qquad \text{or,}\quad \begin{array}{r} 1\cdot884{,}955{,}592 \\ 404 \\ \hline 1\cdot884{,}955{,}188 \end{array}$$

Hence appears the reason of the direction in the former paper to increase the last figure of the contracted divisor by unit when the next figure of the complete divisor would be 5 or more.

The two sets of factors here are:—

$$\begin{array}{ll} 2\times1\cdot570 & \text{and } \tfrac{1}{6}\times1\cdot884 \\ \times1\cdot000{,}507 & \times1\cdot000{,}507 \\ \times1\cdot000{,}000{,}214 & \times1\cdot000{,}000{,}214 \\ \times1\cdot000{,}000{,}000{,}000{,}410; & \times1\cdot000{,}000{,}000{,}410.^{*} \end{array}$$

And the table here requisite is one that shall contain log $(1+\cdot001^{m}n)$ for values of n from 000 to 999, and for values of m from 1 to 4. Practically, it is unnecessary to tabulate the column corresponding to $m=4$, as the values belonging to it are those in the adjoining column with the last period cut off. The table here described is that which accompanied the former paper; and I apply it to the foregoing operations as follows:—

301 029 995 664	log 2	221 848 749 616	colog 6
195 899 652 409	570 Col. 1	275 080 898 457	884
220 131 504	507 „ 2	220 131 504	507
92 939	214 „ 3	92 939	214
178	410 „ 4	178	410
0·497 149 872 694		0·497 149 872 694	

The reader is now in a position to judge of the comparative facility of the three modes of formation which have been exemplified, so far as the number of figures and of tabular entries requisite in each of them is concerned. Obviously the superiority in these respects rests with the last—the three-figure method. But it is in the resolving process that the superiority of this method chiefly resides, and it will soon be felt and admitted if the three modifications of the process are tried. The interruptions to the division for the formation of new divisors, when numerous, are attended by a degree of irksomeness which is not conducive to accuracy. The three-figure process, when applied to the formation of twelve-figure logarithms, has only one of these breaches of continuity, the formation of the divisor corresponding to which also requires hardly a thought; and when applied to the formation of nine-figure logarithms it has no breach at all (see *Ex.* 8, p. 81).

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* See Note, p. 127.

I am now to elucidate the converse operation—that of finding the number corresponding to a given logarithm. The process consists in decomposing the given logarithm by subtraction into a series of other logarithms contained in the tables, by which the factors of the required number become known. Multiplication of these factors then gives the required number. So far no demonstration is needed. It is necessary, however, to show that a single entry in the auxiliary table and in each column of the principal table will always suffice to exhaust the given logarithm; since on this circumstance depends the form of the *compounding* operation, as I call the multiplication of the factors.

The property of the tables just referred to belongs to them in virtue of the constitution of the several columns and their relation to each other. It belongs equally to all the tables, whether adapted to the one-figure, the two-figure, or the three-figure process; but I confine my illustrations to the last—the three-figure table—as the others are not before us.

For the present purpose we have to do with five columns, one —say the first—of the auxiliary table, and the four columns of the principal table. In the first of these columns the least term is log 2, and in each of the others I call that the least term which corresponds to 001. If for a moment we conceive log 10 (or 1) to be the least term in an imaginary column preceding the auxiliary column, then if to each of the five columns before us the least term in that immediately preceding were annexed, it would be found, that in each of the series so formed the difference between any two consecutive terms is less than the least term in that series. So that if from a logarithm intermediate in value to two consecutive terms in any of the series, the less of the two were subtracted, the remainder, being necessarily less than the difference between the two, would also be less than the least term in that series. Hence, generally, if with any given logarithm less than log 10, we enter the column which contains the greatest value not exceeding it, and subtract from it that value; if with the remainder we enter in like manner a subsequent column, subtracting as before, and so on, till we reach the last column, we should, if the logarithm have not been exhausted before, succeed in exhausting it by an entry in that column, since it contains all values from the least in Column III. to 0. And thus, with in no case more than a single entry in each column, the given logarithm will be decomposed into a series of tabular logarithms; and the numbers corresponding to these being known, the factors of the required number are determined.

An entry in the auxiliary table is *necessary* only if the logarithm to be decomposed be not less than log 2; but we may, nevertheless, if we please, *always* commence with an entry in that table. For it will be found that, in the case of *any* given logarithm, the auxiliary table affords a choice of no fewer than five or six distinct logarithms, the subtraction of any one of which (indices being neglected), will leave a remainder less than log 2. The advantage of this is, that we have always the means of verifying our results by employing different methods of decomposition.

It is, as will be seen presently, the argument figures, corresponding to the logarithms into which any given logarithm is decomposed, that form the multipliers in the compounding process. Hence, the component logarithms being taken from successive columns (no two of them from the same column), it follows, from the relation of the values in the several columns to their corresponding arguments, that the multipliers descend in orderly succession from the decimal point: no two of them—of how many figures soever each may consist—can, so to speak, *overlap* each other. It is by the knowledge of this that we are guided in the arrangement of the compounding process.

I shall now give some examples of the formation of antilogarithms, illustrating, as before, the one, two, and three-figure processes.

Given $\log \pi = 0{\cdot}497149872694$; required π.

497149872694	log 2	1·5	4
301029995664		60	
196119877030	5 Col. 1	1·560	6
176091259056		9360	
20028617974	4 „ 2	1·569360	9
17033339299		14124240	
2995278675	6 „ 3	$1{\cdot}5707724\cancel{2}40$	1
2597980720		15707724	
397297955	9 „ 4	$1{\cdot}570788\cancel{1}31724$	5
		7853941	
		$1{\cdot}57079\cancel{6}985665$	2

397297955	9 Col. 4
390689250	
6608705	1 „ 5
4342923	
2265782	5 „ 6
2171467	
94315	2 „ 7
86859	
7456	1 „ 8
4343	
3113	7 „ 9
3040	
73	1 „ 10
43	
30	7 „ 11
30	

1·57079~~6~~985665	2
314159	1
15708	7
10996	1
157	7
110	
1·570796326795	×2
3·141592653590	$=\pi$

The operation is as above, being the converse of one of those on page 125. The portion on the left shows the decomposition of the given logarithm into the following sum:—

$$\log 2+\log 1{\cdot}5+\log 1{\cdot}04+\log 1{\cdot}006+\log 1{\cdot}0009, \text{ and so on;}$$

and the portion on the right shows the compounding or multiplication of the corresponding factors. It is this latter portion alone that stands in need of explanatory remark.

The factors may, of course, be multiplied in any order, and we might have commenced with 2—the factor corresponding to the logarithm taken from the auxiliary table.* I prefer, generally, to leave this factor to the last, and to use up, first, those of the form $1+n$, and in the order in which they arise. Now, to multiply by a factor of this form, it suffices to add to the multiplicand its product by n; for $a(1+n)=a+an$. So, taking the factor 1·5 as the first multiplicand, we obtain the product of the first two factors by adding to this multiplicand its product by ·04, and thus form a second multiplicand. In like manner, using ·006 as a multiplier, we obtain the product of the first three factors, and so form a third multiplicand, and so on.

But it is convenient to disregard the ciphers which precede the significant figures in the several multipliers, as we can determine by other indications the position in the decimal scale which the several partial products ought to occupy. We know that the significant figures of the several multipliers descend regularly in the scale, and in the one-figure process the exact place of each corresponds with the number of the column placed against it. Thus, 5 is in the first decimal place, 4 in the second, and so on. Consequently, the last figure of the first multiplicand being in the first place, and the multiplier being in the second, the right-hand figure of their product will be in the third, since $1+2=3$. In like manner, the last figure of the second multiplicand being in the third place, and the corresponding multiplier being also in the third, the right-hand figure of their product will be in the sixth, since $3+3=6$. This principle will guide us as to the placing of the next partial product, the last figure of which will fall in the tenth place, since $6+4=10$. The following product, however, would extend beyond the twelfth place, which is the limit of our operation, and therefore we here apply the principle somewhat differently. The next multiplier, 1, is in the fifth place, and therefore, since $12-5=7$, it follows that the seventh decimal place of the multiplicand is that whose product will fall in the twelfth place. Contraction, therefore, commences at this point, and it will readily appear that each succeeding multiplicand will have one effective figure fewer than that which preceded it. After the formation of the sixth multiplicand, the operation passes into that of contracted multiplication, a figure being cut off from the multiplicand at each step.

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* See Example, p. 90.

Another and somewhat different method may be employed to ensure the proper placing of the products in the compounding operation. The first portion of the operation in question is repeated in the margin, to exemplify the process now referred to. The decimal places of the first multiplicand being made up to twelve by the annexation of ciphers, the orderly curtailment of it and the succeeding multiplicands in the manner shown, furnishes, it will readily be perceived, a very perspicuous guide for the properly placing of the several partial products. This method will probably be found superior to the other when paper ruled in squares is not used.

$$\begin{array}{r|l}
1{\cdot}5000000000\cancel{0}0 & 4 \\
60000000000 & \\ \hline
1{\cdot}560000000\cancel{0}00 & 6 \\
9360000000 & \\ \hline
1{\cdot}56936000\cancel{0}000 & 9 \\
1412424000 & \\ \hline
1{\cdot}5707724\cancel{4}000 & 1 \\
15707724 & \\ \hline
1{\cdot}570788\cancel{1}31724 & 5 \\
7853941 & \\ \hline
1{\cdot}57079\cancel{5}985665 &
\end{array}$$

I now propose the same example by the two-figure process :—

49 71 49 87 26 94	colog 6	1·88	26
22 18 48 74 96 16		37 6	
		11 28	
27 53 01 12 30 78	88 Col. 1		
27 41 57 84 92 64		1·88 48 88	35
		56 54 66 4	
11 43 27 38 14	26 „ 2	9 42 44 40	
11 27 70 02 77			
		1·88 49 5~~3~~ 97 10 80	86
15 57 35 37	35 „ 3	1 50 79 63	
15 20 00 41		11 30 97	00 07
		13	
37 34 96	86 „ 4		
37 34 93		1·88 49 55 59 21 53	÷6
3	07 „ 6	3·14 15 92 65 35 88	=π
3			

Above are the two portions of the operation, the second only of which, as in the former case, needs remark. The multipliers consist each of two figures, but the principle by which we were guided in placing the products in the previous example is equally applicable here. The last figure of the first multiplicand is in the second decimal place, and the first figure of the multiplier, 26, is in the third. Therefore, since 2 + 3 = 5, the right-hand figure of the first product will fall in the fifth place, and that of the second product will fall in the sixth. In like manner, the right-hand figures of the next two products will fall in the eleventh and twefth places respectively (6 + 5 = 11, and 6 + 6 = 12). The first figure of the third multiplier, 86, is in the seventh place; and hence, since 12 − 7 = 5, the fifth decimal place of the multiplicand will be the last effective figure within our limit of twelve places. The process is now that of contracted multiplication, and the absence of a factor from Col. 5 will give no trouble.

Finally, I repeat the example by the three-figure process: —

497 149 872 694	log 2	1·570	507
301 029 995 664		785 0	
		10 990	
196 119 877 030	570 Col. 1		
195 899 652 409		1·570 79~~6~~ 990	214
		314 159	
220 224 621	507 „ 2	15 708	
220 131 504		6 283	410
93 117	214 „ 3	628	
92 939		16	
178	410 „ 4	1·570 796 326 794	×2
178		3·141 592 653 588	

The same principle is our guide in the placing of the products here. For the first product we have $3+4=7$, and for the second $3+6=9$, the latter falling *two* places in advance of the former, in consequence of the occurrence of a cipher. For the next we have $12-7=5$; that is, the fifth decimal place of the second multiplicand is the last effective figure, its product falling in the twelfth place. And here, also, the operation changes into that of contracted multiplication.

It is with a view to perspicuity that I have exemplified the three processes, as each of the last two follows from that which precedes it by a natural sequence. But it will be understood that it is the tables for the application of the three-figure process alone that accompany these papers. The superiority of the process last named over the others, in regard to the number of tabular entries and of figures employed, is equally apparent in the anti-logarithmic as in the logarithmic application. It will be still more fully appreciated after the trial of a few examples.

I must postpone till another opportunity a description of the methods employed for the construction and verification of the tables, and also an account of the origin of the processes that have been exemplified.

On Interpolation, Summation, and the Adjustment of Numerical Tables. (Part III.) By W. S. B. Woolhouse, F.R.A.S., F.S.S., *Vice-President of the Institute of Actuaries, &c.*

[Read before the Institute, 27th February, 1865.]

THE ADJUSTMENT OF NUMERICAL TABLES.

WHEN a series of quantities which depend on a fixed law, whether known or implied, follow each other in a due order of succession, the general accuracy of their numerical values may be satisfactorily tested by observing the regularity of the progression of a suitable order of differences. If the tabular quantities are the results of calculation from a given formula, with equidistant arguments, by differencing them up to a certain order the existence of an isolated error, if one should exist, is thus prominently exhibited and therefore speedily detected. Moreover the precise locality of the error and the fact of its individuality being indicated by a characteristic interruption of the law of progression, the proper correction is ascertained by carefully noting the central position of the dis-